Available online at www.sciencedirect.com

ScienceDirect

ELSEVIER

J. Differential Equations 233 (2007) 1–21

Journal of Differential Equations

www.elsevier.com/locate/jde

Generalized solvability and optimization of a parabolic system with a discontinuous solution

D. Nomirovskii [a,b]

[a] *National Taras Shevchenko University of Kyiv, Ukraine*
[b] *Department of Cybernetics, Kyiv University, Volodymyrska, 64 01033 Kyiv, Ukraine*

Received 12 November 2004; revised 17 August 2006

Available online 9 November 2006

Abstract

We consider the following linear parabolic system in a domain with a thin low-permeable insertion ("imperfect interface"):

$$\frac{\partial u}{\partial t} + q(\xi)u + \nabla \cdot \vec{\omega} = f(t,\xi), \qquad \vec{\omega} = -\mathbf{K}\nabla u, \qquad (t,\xi) \in Q_1 \cup Q_2 \subset \mathbb{R}^n,$$
$$u|_{t=0} = 0, \qquad u|_{\xi \in \partial\Omega} = 0, \qquad \mathbf{K} = \{k_{ij}(\xi)\}_{i,j=1}^n,$$
$$\big[(\vec{\omega}, \vec{n})_{\mathbb{R}^n}\big] = 0, \qquad \alpha[u] + \lim_{\xi \to \xi_0} (\vec{\omega}, \vec{n})_{\mathbb{R}^n} = 0, \qquad (t, \xi_0) \in Q_3 = \bar{Q}_1 \cap \bar{Q}_2.$$

We consider a new formulation of the problem where the unknowns are $(u, \vec{\omega})$, and the parabolic problem is converted to a first-order system of partial differential equations with distributional coefficients. We also prove inequalities for negative norms for the parabolic operator with the distributional coefficients and theorems of existence and uniqueness. For optimization problems for the processes we show existence of optimal controls, investigate smoothness of a performance criterion and give a simple condition for controllability of the system. In addition, we consider applications of the obtained results to a pulse control problem and prove convergence of a control mapping regularization procedure.

Keywords: Linear equation; Parabolic equation; Discontinuous solution; Transmission problem; Solvability; Optimization; Distribution; Pulse control

E-mail address: kashpir@mail.ru.

0022-0396/$ – see front matter
doi:10.1016/j.jde.2006.09.025

1. Introduction

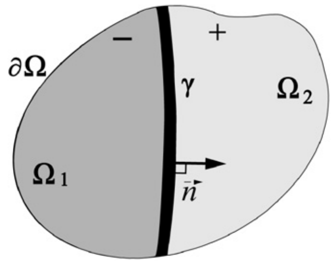
There are many actual physical processes occurring in media with foreign zones and insertions. Heat and mass transmission problems often occur in domains with thin low-permeable insertions: paint layers, refractories, gas gaps, thin liquid layers, laminas, cracks, edges of metal granules, etc. When such transmission problems are studied, the foreign zone is eliminated from the domain where heat and mass transmission takes place, and interface (transmission) conditions on the surfaces of insertions are established. Thus, one gets a boundary-value problem in a disconnected domain. There are many papers concerning these problems [1–32], but many problems of solvability and optimization of parabolic systems with discontinuous solutions are still open. This transmission problem admits different standard formulations as evolution variational equality, as Banach-valued time-dependent equation, etc., and there are many references concerning problems of this kind.

Another approach to investigation of heat and mass transmission problems in a domain with thin low-permeable insertions is to replace the original partial differential equation and interface conditions by several first-order partial differential equations which account for the interface conditions themselves [26–30]. In this method, the eliminated insertion is returned to the domain of transmission again, the general equation and transmission conditions turn into a system of first-order partial differential equations, but the coefficients of the equations are now distributions.

In this paper we consider the above approach to the problem. This formulation has some advantages in comparison with the previous ones. In the first-order system, the roles of the variables ξ and t are symmetric. Presence of several equations in the system leaves much more freedom to prove necessary inequalities concerning the operator than there was available in the initial and direct equations. The first-order partial differential equations have simple physical interpretations (they are generalizations of two physical laws: the conservation law and the law of transportation), so that the system is more appropriate for simulating physical processes. In contrast to the formulations as evolution variational equality, where the unknown function u is from a certain space $L_2((0,T);V)$ for V a Banach space, the formulation as the system of equations allows one to study the time-singular processes from the point of view of distribution theory. In particular, this approach is perfectly suited for studying problems of pulse optimal control, and for solving the problem approximately by mixed finite element methods [33,34]. In addition, under this approach, the domain of the process is simply connected (as opposed to those in traditional formulations), which is of importance for some problems (for example, for numerical procedures).

2. Basic definitions

Let the state function $u(t,\xi)$ be defined in a cylindrical domain $Q=(0,T)\times\Omega$, where $t\in(0,T)$, $\xi=(\xi_1,\dots,\xi_n)\in\Omega=\Omega_1\cup\gamma\cup\Omega_2\subset\mathbb{R}^n$, Ω is a bounded simply connected domain with a regular boundary $\partial\Omega$, and $\bar{\gamma}=\bar{\Omega}_1\cap\bar{\Omega}_2\subset\mathbb{R}^n$ is a smooth surface that divides the domain Ω into two simply connected domains Ω_1 and Ω_2 ($\Omega_1\cap\Omega_2=\emptyset$). Denote $Q_i=(0,T)\times\Omega_i$, for $i\in\{1,2\}$, and $Q_3=(0,T)\times\gamma$.



Let us consider heat and mass transmission that occurs in the two heterogeneous domains Q_1, Q_2 separated by a thin insertion Q_3:

$$\frac{\partial u}{\partial t} + q(\xi)u - \sum_{i,j=1}^{n} \frac{\partial}{\partial \xi_i}\left(k_{ij}(\xi)\frac{\partial u}{\partial \xi_j}\right) = f(t,\xi), \quad (t,\xi) \in Q_1 \cup Q_2, \tag{1}$$

$$u|_{t=0} = 0, \qquad u|_{\xi \in \partial\Omega} = 0. \tag{2}$$

For parabolic equation (1) the following jump conditions

$$\left[(\vec{\omega},\vec{n})_{\mathbb{R}^n}\right] = 0, \qquad \alpha[u] + \lim_{\xi\to\xi_0} (\vec{\omega},\vec{n})_{\mathbb{R}^n} = 0, \qquad (t,\xi_0) \in Q_3 \quad \text{(non-ideal contact)}, \tag{3}$$

$$\left[(\vec{\omega},\vec{n})_{\mathbb{R}^n}\right] = 0, \qquad [u] = 0, \qquad (t,\xi_0) \in Q_3 \quad \text{(ideal contact)}, \tag{4}$$

$$\left[(\vec{\omega},\vec{n})_{\mathbb{R}^n}\right] = g, \qquad [u] = 0, \qquad (t,\xi_0) \in Q_3 \quad \text{(external source)}, \tag{5}$$

$$\left[(\vec{\omega},\vec{n})_{\mathbb{R}^n}\right] + \alpha \lim_{\xi\to\xi_0} u = 0, \qquad [u] = 0, \qquad (t,\xi_0) \in Q_3 \quad \text{(proper source)}, \tag{6}$$

$$\left[(\vec{\omega},\vec{n})_{\mathbb{R}^n}\right] + \beta \lim_{\xi\to\xi_0} u_t = 0, \qquad [u] = 0, \qquad (t,\xi_0) \in Q_3 \quad \text{(lumped heat capacity)}, \tag{7}$$

are often found in the literature [23–32,35]. Here $\vec{\omega} = -\mathbf{K}\,\mathrm{grad}\,u$ in $(t,\xi) \in Q_1 \cup Q_2$, $\mathbf{K} = \{k_{ij}\}_{i,j=1}^{n}$, $\mathrm{grad}\,u = (u_{\xi_1},\ldots,u_{\xi_n})$, and $[u]$ denotes a discontinuous jump of $u(t,\xi)$ on Q_3, i.e.

$$[u](t,\xi_0) = \lim_{\xi^+\to\xi_0} u\left(t,\xi^+\right) - \lim_{\xi^-\to\xi_0} u\left(t,\xi^-\right), \quad \xi^+ \in \Omega_2, \quad \xi^- \in \Omega_1, \quad \xi_0 \in \gamma.$$

In what follows $\alpha(\xi), \beta(\xi) > 0$ are continuous functions in $\xi \in \bar{\gamma}$, $\vec{n} = (n_{\xi_1},\ldots,n_{\xi_n})$ is the normal vector to the surface γ (external to the domain Ω_1), and functions $q(\xi)$, $k_{ij}(\xi)$ have discontinuous jumps on the surface Q_3. However, other configurations of the physical domains $\Omega_1, \Omega_2, \gamma$ are also encountered (for instance, the interface $\bar{\gamma} = \bar{\Omega}_1 \cap \bar{\Omega}_2$ is a closed surface lying strictly inside Ω).

In this paper we consider conditions (3), but the investigation could be adapted for the other jump conditions (4)–(7) and for other geometric configurations of the interface. Jump conditions (3) simulate heat (or mass) transmission throughout the thin foreign insertion with a small overall heat transfer coefficient (low-permeable insertion). Taking into account insertion thickness, one can assume the function u to be continuous and linear along the foreign insertion. Considering the problem under these assumptions and taking the limit as the insertion thickness becomes vanishingly small, one can derive conditions (3), where the function $\alpha(\xi)$ describes physical properties of the insertion.

Let us proceed from (1)–(3) to the set of the linear first order partial differential equations with respect to $(u, \vec{\omega})$.

Let $C^k(\bar{Q}_1, \bar{Q}_2)$ be the set of functions from $C^k(Q_1 \cup Q_2)$ which admit an extension (with the same smoothness) from Q_1 to $\bar{Q}_1$ and from Q_2 to $\bar{Q}_2$, and $C^1_{bd}(\bar{Q}_1, \bar{Q}_2)$ be the set of functions from $C^1(\bar{Q}_1, \bar{Q}_2)$ that satisfy the initial and boundary conditions (2).

Similarly, let $C^1_{bd*}(\bar{Q}_1, \bar{Q}_2)$ be the set of functions from $C^1(\bar{Q}_1, \bar{Q}_2)$ which satisfy the following adjoint conditions

$$v|_{t=T} = 0, \qquad v|_{\xi \in \partial\Omega} = 0. \tag{8}$$

Let C_{bd} be a set of pairs of functions $x = (u, \vec{\omega}) \in C^1_{bd}(\bar{Q}_1, \bar{Q}_2) \times (C(\bar{Q}))^n$ that satisfy conditions (3), and C_{bd*} be a set of pairs $y = (v, \vec{\eta}) \in C^1_{bd*}(\bar{Q}_1, \bar{Q}_2) \times (C(\bar{Q}))^n$ that satisfy condition $\alpha[v] - (\vec{\eta}, \vec{n})_{\mathbb{R}^n} = 0$ on Q_3.

Let $W_2^{1,1/1}(Q)$ be the completion of $C^1_{bd}(\bar{Q}_1, \bar{Q}_2)$ in the norm

$$\|u\|^2_{W_2^{1,1/1}(Q)} = \sum_{k=1}^{2} \int_{Q_k} u_t^2 + \sum_{i=1}^{n} u_{\xi_i}^2 \, dQ_k, \tag{9}$$

and $W_2^{1,1}(Q)$ be the completion of $C^1(\bar{Q})$ in the same norm (9).

It is clear that an element of the space $W_2^{1,1/1}(Q)$ can be thought of as a pair of functions $(u_1, u_2) \in W_2^{1,1}(Q_1) \times W_2^{1,1}(Q_2)$ which satisfy conditions (2) on the corresponding parts of the boundary.

Similarly, let $W_{2,*}^{1,1/1}(Q)$ be the completion of $C^1_{bd*}(\bar{Q}_1, \bar{Q}_2)$ in the norm (9). Denote by $W_2^{-1,1/1}(Q)$, $W_{2,*}^{-1,1/1}(Q)$ the conjugate spaces to $W_2^{1,1/1}(Q)$, $W_{2,*}^{1,1/1}(Q)$, respectively.

Due to the theorem on traces, functions from $(u_1, u_2) \in W_2^{1,1}(Q_1) \times W_2^{1,1}(Q_2)$ track $(u^-, u^+) \in L_2(Q_3) \times L_2(Q_3)$ on the surface Q_3 and the trace operator is continuous. Hence there exists a positive constant $c > 0$ such that for all $u \in W_2^{1,1/1}(Q)$ the following inequality holds:

$$\int_{Q_3} [u]^2 \, dQ_3 \leqslant c\|u\|^2_{W_2^{1,1/1}(Q)}, \tag{10}$$

where discontinuous jump $[u]$ of the function $u(t, \xi)$ on Q_3 is understood in the sense of trace theory.

Analogously, we can prove the inequality

$$\big\|[v]\big\|_{L_2(Q_3)} \leqslant c\|v\|_{W_{2,*}^{1,1/1}(Q)}$$

for all $v \in W_{2,*}^{1,1/1}(Q)$.

Introduce the completion X (respectively Y) of C_{bd} (respectively C_{bd*}) in the norm

$$\|x\|^2 = \|u\|^2_{W_2^{1,1/1}(Q)} + \|\vec{\omega}\|^2_{L_2^n(Q)}.$$

Note that the vector $\vec{\omega}$ from a pair of functions $x=(u,\vec{\omega})\in X$ tracks $(\vec{\omega},\vec{n})_{\mathbb{R}^n}$ on the surface Q_3 defined by equality $(\vec{\omega},\vec{n})_{\mathbb{R}^n}=-\alpha[u]$. Likewise, the vector $\vec{\eta}$ tracks $(\vec{\eta},\vec{n})_{\mathbb{R}^n}=\alpha[v]$ in a couple $y=(v,\vec{\eta})\in Y$ on Q_3.

The relation between the vector $\vec{\omega}$ and its trace $(\vec{\omega},\vec{n})_{\mathbb{R}^n}$ on Q_3 becomes clear after considering the norm

$$\|x\|^2=\|u\|^2_{W_2^{1,1/1}(Q)}+\|\vec{\omega}\|^2_{L_2^n(Q)}+\big\|(\vec{\omega},\vec{n})_{\mathbb{R}^n}\big\|^2_{L_2(Q_3)},$$

on the set C_{bd}. Indeed, this norm is equivalent to the norm of the space X, and if $L^n_{2,\gamma}(Q)$ is the completion of $(C(\bar{Q}))^n$ in the following norm

$$\|\vec{\omega}\|^2_{L^n_{2,\gamma}(Q)}=\|\vec{\omega}\|^2_{L_2^n(Q)}+\big\|(\vec{\omega},\vec{n})_{\mathbb{R}^n}\big\|^2_{L_2(Q_3)},$$

then an element of the space $L^n_{2,\gamma}(Q)$ is a function $\vec{\omega}$ from $L_2^n(Q)$, and the trace $(\vec{\omega},\vec{n})_{\mathbb{R}^n}\in L_2(Q_3)$ makes sense.

More precisely, the space $L^n_{2,\gamma}(Q)$ is isometric to $L_2^n(Q)\times L_2(Q_3)$, and isometry operator $O: L^n_{2,\gamma}(Q)\to L_2^n(Q)\times L_2(Q_3)$ is defined as the continuous extension of the operator $(C(\bar{Q}))^n\ni\vec{\omega}\to O\vec{\omega}=(\vec{\omega},(\vec{\omega},\vec{n})_{\mathbb{R}^n})\in L_2^n(Q)\times L_2(Q_3)$ to the whole space $L^n_{2,\gamma}(Q)$.

A natural bilinear form $\langle\cdot,\cdot\rangle_{X\times X^*}$ is defined on the Cartesian products of the initial spaces and their conjugate spaces (for instance, X and X^*).

Consider a system which describes heat and mass transmission in two domains with thin low-permeable insertions

$$\mathcal{L}x=F, \tag{11}$$

with the operator $\mathcal{L}$ defined by the following symbolic matrix:

$$\mathcal{L}=\left(\begin{array}{c|c} \frac{\partial}{\partial t}+q & \left(\frac{\partial}{\partial\xi_1},\dots,\frac{\partial}{\partial\xi_n}\right) \\ \hline \begin{pmatrix}\frac{\partial}{\partial\xi_1}\\ \vdots\\ \frac{\partial}{\partial\xi_n}\end{pmatrix} & \mathbf{M}\end{array}\right),\qquad x=\begin{pmatrix}u\\ \vec{\omega}\end{pmatrix}.$$

Here the function $u(t,\xi)$ describes heat and mass transmission, and $\vec{\omega}=(\omega_1,\dots,\omega_n)$ is a vector of heat flux.

The operator $\mathcal{L}: X\to Y^*$ maps its domain of definition $D(\mathcal{L})=C_{bd}$ from space X to space Y^*.

The coefficient matrix $\mathbf{M}=\{\sigma_{ij}\}_{i,j=1}^n$ can be presented as $\mathbf{M}=\mathbf{K}^{-1}+\alpha^{-1}\delta(\gamma)\mathbf{P}$, and the coefficients of the system satisfy the conditions $q(\xi)\in C(\bar{\Omega}_1,\bar{\Omega}_2)$, $q\geqslant 0$, $\bar{k}_{ij}(\xi)\in C(\bar{\Omega}_1,\bar{\Omega}_2)$, where $\mathbf{K}^{-1}=\{\bar{k}_{ij}\}_{i,j=1}^n$ is the inverse to the matrix of coefficients $\mathbf{K}=\{k_{ij}\}_{i,j=1}^n$ of the initial parabolic equation (which is assumed to be symmetric and positive definite). Here $\delta(\gamma)$ is the delta function, and $\mathbf{P}=\{p_{ij}\}_{i,j=1}^n$ is the projection matrix to the normal $\vec{n}$ of the surface γ; $p_{ij}=n_{\xi_i}n_{\xi_j}$. Thus, we have $\sigma_{ij}=\bar{k}_{ij}+\alpha^{-1}\delta(\gamma)n_{\xi_i}n_{\xi_j}$.

We give below some motivation as to why the matrix $\mathbf{M} = \mathbf{K}^{-1} + \alpha^{-1}\delta(\gamma)\mathbf{P}$ should be applied. As is well known, the equality

$$\operatorname{grad} u = \operatorname{grad}_{\mathrm{cl}} u + [u]\vec{n} \tag{12}$$

is valid for the operator $\operatorname{grad} u$ in distribution theory, where $\operatorname{grad}_{\mathrm{cl}} u$ is a standard differential operator in the classical sense. By considering $\operatorname{grad}_{\mathrm{cl}} u = \mathbf{K}^{-1}\vec{\omega}$ and jump conditions (3), equality (12) can be rewritten in the form

$$\operatorname{grad} u + \mathbf{K}^{-1}\vec{\omega} + \frac{1}{\alpha}(\vec{\omega}, \vec{n})_{\mathbb{R}^n}\vec{n} = 0$$

or

$$\operatorname{grad} u + \big(\mathbf{K}^{-1} + \alpha^{-1}\delta(\gamma)\mathbf{P}\big)\vec{\omega} = 0,$$

as required in (11).

Thus, in Eq. (11) we can denote by $\operatorname{div}\vec{\omega} = (\omega_1)_{\xi_1} + \cdots + (\omega_n)_{\xi_n}$ the continuous linear functional defined at every function $v \in W_{2,*}^{1,1/1}$ as

$$\langle \operatorname{div}\vec{\omega}, v\rangle_{W_{2,*}^{-1,1/1}\times W_{2,*}^{1,1/1}} = -\sum_{k=1}^{2}\sum_{i=1}^{n}\int_{Q_k}\omega_i\frac{\partial v}{\partial \xi_i}\,dQ_k - \int_{Q_3}(\vec{\omega}, \vec{n})_{\mathbb{R}^n}[v]\,dQ_3.$$

Note that for smooth functions the above equality is equivalent to the partial integration formula.

By $\operatorname{grad} u$ we denote the continuous linear functional defined at $\vec{\eta} \in L_{2,\gamma}^n(Q)$ or at $y \in Y$ as

$$\langle \operatorname{grad} u, \vec{\eta}\rangle_{(L_{2,\gamma}^n)^*\times L_{2,\gamma}^n} = \sum_{k=1}^{2}\sum_{i=1}^{n}\int_{Q_k}\frac{\partial u}{\partial \xi_i}\eta_i\,dQ_k + \int_{Q_3}[u](\vec{\eta}, \vec{n})_{\mathbb{R}^n}\,dQ_3.$$

By $\mathbf{M}\vec{\omega}$ we also mean the continuous linear functional defined at $\vec{\eta} \in L_{2,\gamma}^n(Q)$ as

$$\langle \mathbf{M}\vec{\omega}, \vec{\eta}\rangle_{(L_{2,\gamma}^n)^*\times L_{2,\gamma}^n} = \sum_{k=1}^{2}\sum_{i,j=1}^{n}\int_{Q_k}\bar{k}_{ij}\omega_j\eta_i\,dQ_k + \int_{Q_3}\alpha^{-1}(\vec{\omega}, \vec{n})_{\mathbb{R}^n}(\vec{\eta}, \vec{n})_{\mathbb{R}^n}\,dQ_3.$$

Taking into account conditions (3), we have

$$\begin{aligned}\langle \mathcal{L}x, y\rangle_{Y^*\times Y} = {} & \sum_{k=1}^{2}\int_{Q_k}\frac{\partial u}{\partial t}v + quv + \sum_{i,j=1}^{n}\bar{k}_{ij}\omega_j\eta_i\,dQ_k \\ & + \sum_{k=1}^{2}\sum_{i=1}^{n}\int_{Q_k}\frac{\partial u}{\partial \xi_i}\eta_i - \frac{\partial v}{\partial \xi_i}\omega_i\,dQ_k + \int_{Q_3}\alpha[u][v]\,dQ_3.\end{aligned} \tag{13}$$

By $\mathcal{L}^+$ we denote the adjoint operator

$$\mathcal{L}^+ y = G, \qquad \mathcal{L}^+ : Y \to X^*, \quad y = (v, \vec{\eta}).$$

Below is the symbolic matrix of the operator $\mathcal{L}^+$:

$$\mathcal{L}^+ = \left(\begin{array}{c|c} -\frac{\partial}{\partial t} + q & \left(-\frac{\partial}{\partial \xi_1}, \ldots, -\frac{\partial}{\partial \xi_n}\right) \\ \hline \begin{pmatrix} -\frac{\partial}{\partial \xi_1} \\ \vdots \\ -\frac{\partial}{\partial \xi_n} \end{pmatrix} & \mathbf{M} \end{array}\right),$$

and coefficients of this matrix are defined here in the same way as for $\mathcal{L}$.

Because the set $D(\mathcal{L}^+) = C_{bd*}$ is the domain of the operator $\mathcal{L}^+$, we have

$$\begin{aligned} \langle \mathcal{L}^+ y, x \rangle_{X^* \times X} &= \sum_{k=1}^{2} \int_{Q_k} -\frac{\partial v}{\partial t} u + quv + \sum_{i,j=1}^{n} \bar{k}_{ij} \omega_j \eta_i \, dQ_k \\ &\quad + \sum_{k=1}^{2} \sum_{i=1}^{n} \int_{Q_k} \frac{\partial u}{\partial \xi_i} \eta_i - \frac{\partial v}{\partial \xi_i} \omega_i \, dQ_k + \int_{Q_3} \alpha[v][u] \, dQ_3 \\ &= \langle y, \mathcal{L} x \rangle_{Y \times Y^*}, \end{aligned} \tag{14}$$

for all $x \in D(\mathcal{L})$, $y \in D(\mathcal{L}^+)$.

3. Properties of the problem

Applying equalities (13) and (14), it is easy to show that the operators $\mathcal{L}$ and $\mathcal{L}^+$ are continuous in their domains of definition.

Due to the density of $D(\mathcal{L})$ in X (respectively, $D(\mathcal{L}^+)$ in Y), there exists the continuous extension of $\mathcal{L}$ (respectively, $\mathcal{L}^+$) to the whole space X (respectively, Y). Let $\bar{\mathcal{L}}$ and $\bar{\mathcal{L}}^+$ denote extended operators.

The following lemma holds:

Lemma 1. *There exists a positive constant $c > 0$ such that for all $x \in X$ and $y \in Y$ the following inequalities hold*:

$$\|\bar{\mathcal{L}} x\|_{Y^*} \leqslant c \|x\|_X, \qquad \left\|\bar{\mathcal{L}}^+ y\right\|_{X^*} \leqslant c \|y\|_Y. \tag{15}$$

Remark 1. Passing to the limit in equality (14) and applying inequality (15), we conclude that operators $\bar{\mathcal{L}}$, $\bar{\mathcal{L}}^+$ satisfy the equality:

$$\langle \bar{\mathcal{L}} x, y \rangle_{Y^* \times Y} = \langle x, \bar{\mathcal{L}}^+ y \rangle_{X \times X^*}, \quad \forall x \in X, \ y \in Y.$$

The relation between a solution of the equation $\bar{\mathcal{L}} x = F$ and a classical solution of problem (1)–(3) is given by the following theorem:

Theorem 1. *Let coefficients k_{ij} of operator $\bar{\mathcal{L}}$ and solution $x = (u, \vec{\omega}) \in X$ of equation $\bar{\mathcal{L}}x = (f, \vec{0}) \in Y^*$, $f \in C(Q_1 \cup Q_2)$ be smooth enough for the classical statement of equations* (1)–(3), *namely*:

(1) $u_t \in C(Q_1 \cup Q_2)$, $u_{\xi_i \xi_j} \in C(Q_1 \cup Q_2)$, $k_{ij} \in C^1(\Omega_1 \cup \Omega_2)$, $1 \leqslant i, j \leqslant n$;
(2) *the following limits exist*: $\lim_{t \to 0} u(t, \xi)$, $\lim_{\xi_k \to \partial \Omega_k} u(t, \xi)$, $\lim_{\xi_k \to \gamma} (\mathbf{K} \operatorname{grad} u, \vec{n})_{\mathbb{R}^n}$, $\xi_k \in \Omega_k$, $k \in \{1, 2\}$.

Then the function $u(t, \xi)$ satisfies equalities (1)–(3) *at every point.*

Proof. Since $x = (u, \vec{\omega}) \in X$, and norm of $W_2^{1,1/1}(Q)$ retains the corresponding limiting values of u, conditions (2) are satisfied.

On the other hand, for all $y = (v, \vec{\eta}) \in Y$ the following equality is valid:

$$\langle \bar{\mathcal{L}}x, y \rangle_{Y^* \times Y} = \langle F, y \rangle_{Y^* \times Y} = (f, v)_{L_2(Q) \times L_2(Q)}. \tag{16}$$

Consider $y = (0, \vec{\eta}) \in Y$. In this case, using (13), we can rewrite equality (16) as

$$\langle \bar{\mathcal{L}}x, y \rangle_{Y^* \times Y} = \sum_{k=1}^{2} \sum_{i=1}^{n} \int_{Q_k} \sum_{j=1}^{n} \bar{k}_{ij} \omega_j \eta_i + u_{\xi_i} \eta_i \, dQ_k = 0, \quad \forall \eta_i \in L_2(Q).$$

Hence, $\vec{\omega} = -\mathbf{K} \operatorname{grad} u$ on $L_2(Q)$, and therefore at every point $(t, \xi) \in Q_1 \cup Q_2$, due to the smoothness of $u(t, \xi)$.

Substitute $y = (v, \vec{\eta}) \in Y$, $v \in C^1(Q)$ and $v = 0$ in Q_3 into (16). Then $[v] = 0$ in Q_3. Integrating by parts, we have

$$\langle \bar{\mathcal{L}}x, y \rangle_{Y^* \times Y} = \sum_{k=1}^{2} \int_{Q_k} \frac{\partial u}{\partial t} v + quv + \sum_{i=1}^{n} \frac{\partial \omega_i}{\partial \xi_i} v \, dQ_k = (f, v)_{L_2(Q) \times L_2(Q)}.$$

Since the set of functions $v(t, \xi)$ is dense everywhere in $L_2(Q)$, we obtain that

$$\frac{\partial u}{\partial t} + qu + \sum_{i=1}^{n} \frac{\partial \omega_i}{\partial \xi_i} = \frac{\partial u}{\partial t} + qu - \sum_{i,j=1}^{n} \frac{\partial}{\partial \xi_i} \left(k_{ij} \frac{\partial u}{\partial \xi_j} \right) = f$$

in $L_2(Q)$. Thus, f is continuous at every point $(t, \xi) \in Q_1 \cup Q_2$.

Substituting $y = (v, \vec{\eta}) \in Y$, $v \in C^1(\bar{Q}_1)$ and $v = 0$ in Q_2 into (16), we infer that

$$\langle \bar{\mathcal{L}}x, y \rangle_{Y^* \times Y} - (f, v)_{L_2(Q) \times L_2(Q)} = \int_{Q_3} \bigl(\alpha[u] + (\vec{\omega}, \vec{n})_{\mathbb{R}^n}^- \bigr) [v] \, dQ_3 = 0,$$

where $(\vec{\omega}, \vec{n})_{\mathbb{R}^n}^- = \lim_{\xi \to \xi_0} (\vec{\omega}, \vec{n})_{\mathbb{R}^n}$, $\xi \in \Omega_1$, $\xi_0 \in \gamma$. Therefore, $\alpha[u] + (\vec{\omega}, \vec{n})_{\mathbb{R}^n}^- = 0$ in Q_3. In the same manner, we have that $\alpha[u] + (\vec{\omega}, \vec{n})_{\mathbb{R}^n}^+ = 0$. Thus, conditions (3) are satisfied. □

Fix functions $a(t)$ and $b(t)$ in $C^1([0,T])$ which satisfy conditions $a(t)>0$, $a'(t)>0$, $b(t)>0$, $b'(t)<0$ on $[0,T]$.

Let $\|x\|_{X_1}$ be the following semi-norm on X:

$$\|x\|^2_{X_1}=\int\limits_Q u^2\,dQ+\sum_{k=1}^2\sum_{i=1}^n\int\limits_{\Omega_k}\left(\int\limits_0^T bu_{\xi_i}\,d\tau\right)^2 d\Omega_k$$
$$+\sum_{k=1}^2\sum_{i=1}^n\int\limits_{Q_k}\left(\int\limits_T^t bu_{\xi_i}\,d\tau\right)^2 dQ_k+\int\limits_{Q_3}\left(\int\limits_T^t b[u]\,d\tau\right)^2 dQ_3+\int\limits_\gamma\left(\int\limits_0^T b[u]\,d\tau\right)^2 d\gamma,$$

and let $\|y\|_{Y_1}$ be the following semi-norm on Y:

$$\|y\|^2_{Y_1}=\int\limits_Q v^2\,dQ+\sum_{k=1}^2\sum_{i=1}^n\int\limits_{\Omega_k}\left(\int\limits_0^T av_{\xi_i}\,d\tau\right)^2 d\Omega_k$$
$$+\sum_{k=1}^2\sum_{i=1}^n\int\limits_{Q_k}\left(\int\limits_0^t av_{\xi_i}\,d\tau\right)^2 dQ_k+\int\limits_{Q_3}\left(\int\limits_0^t a[v]\,d\tau\right)^2 dQ_3+\int\limits_\gamma\left(\int\limits_0^T a[v]\,d\tau\right)^2 d\gamma.$$

We denote by X_1 the completion of $C^1_{bd}(\bar Q_1,\bar Q_2)$ in the norm $\|u\|_{X_1}$, and by Y_1 the completion of $C^1_{bd*}(\bar Q_1,\bar Q_2)$ in the norm $\|v\|_{Y_1}$. One can easily prove that the dense continuous embeddings $W_2^{1,1/1}(Q)\subset X_1$, $W_{2,*}^{1,1/1}(Q)\subset Y_1$ are valid.

Lemma 2. *There exists a positive constant $c>0$ such that for all $x\in X$ the following inequality holds*:

$$c^{-1}\|x\|_{X_1}\leqslant\|\bar{\mathcal L}x\|_{Y^*}.$$

Proof. Consider the value of the functional $\bar{\mathcal L}x\in Y^*$ at a point $y=Ix\in Y$ where

$$v=-\int\limits_T^t b(\tau)u(\tau,\xi)\,d\tau,\qquad \vec\eta=\mathbf K\operatorname{grad}v.$$

It is clear that $y=Ix\in Y$.

By definition of the operator $\bar{\mathcal L}x$, we have

$$\langle\bar{\mathcal L}x,y\rangle_{Y^*\times Y}=\sum_{k=1}^2(u_t+qu,v)_{L_2(Q_k)}+\sum_{k=1}^2\sum_{i,j=1}^n(\bar k_{ij}\omega_j,\eta_i)_{L_2(Q_k)}$$
$$+\sum_{k=1}^2\sum_{i,j=1}^n(u_{\xi_i},\eta_i)_{L_2(Q_k)}-\sum_{k=1}^2\sum_{i=1}^n(\omega_i,v_{\xi_i})_{L_2(Q_k)}+\big(\alpha[u],[v]\big)_{L_2(Q_3)}.$$

Consider each summand separately. Integrating by parts and using conditions (2), we get

$$
\begin{aligned}
(u_t+qu,v)_{L_2(Q)} &= -(u,v_t)_{L_2(Q)}-\big(qb^{-1}v_t,v\big)_{L_2(Q)}\\
&= \int\limits_Q bu^2\,dQ+\frac12\int\limits_\Omega \frac{q}{b}v^2\Big|_{t=0}\,d\Omega+\int\limits_Q \frac{q(-b')}{b^2}v^2\,dQ\geqslant c^{-1}\|u\|^2_{L_2(Q)}.
\end{aligned}
$$

Next, consider the second summand:

$$
\sum_{k=1}^2\sum_{i,j=1}^n\int\limits_{Q_k}\bar k_{ij}\omega_i\eta_j\,dQ_k=\sum_{k=1}^2\sum_{i=1}^n\int\limits_{Q_k}\omega_i\sum_{j=1}^n\bar k_{ij}\eta_j\,dQ_k=\sum_{k=1}^2\sum_{i=1}^n\int\limits_{Q_k}\omega_i\frac{\partial v}{\partial\xi_i}\,dQ_k.
$$

Now consider the third summand. Since the matrix $\{k_{ij}\}_{i,j=1}^n$ is positive definite, integrating by parts, we obtain

$$
\begin{aligned}
\sum_{k=1}^2\sum_{i=1}^n\int\limits_{Q_k}u_{\xi_i}\eta_i\,dQ_k &= \sum_{k=1}^2\sum_{i,j=1}^n\int\limits_{Q_k}u_{\xi_i}k_{ij}v_{\xi_j}\,dQ_k\\
&= -\sum_{k=1}^2\sum_{i,j=1}^n\int\limits_{Q_k}\frac{k_{ij}}{b}v_{t\xi_i}v_{\xi_j}\,dQ_k\\
&= \sum_{k=1}^2\sum_{i,j=1}^n\int\limits_{\Omega_k}\frac{k_{ij}}{2b}v_{\xi_i}v_{\xi_j}\Big|_{t=0}\,d\Omega_k+\sum_{k=1}^2\sum_{i,j=1}^n\int\limits_{Q_k}\frac{(-b)'k_{ij}}{2b^2}v_{\xi_i}v_{\xi_j}\,dQ_k\\
&\geqslant c^{-1}\sum_{k=1}^2\sum_{i=1}^n\int\limits_{\Omega_k}\Bigg(\int\limits_0^T bu_{\xi_i}\,d\tau\Bigg)^2 d\Omega_k+c^{-1}\sum_{k=1}^2\sum_{i=1}^n\int\limits_{Q_k}\Bigg(\int\limits_T^t bu_{\xi_i}\,d\tau\Bigg)^2 dQ_k.
\end{aligned}
$$

Finally, we discuss the last summand.

$$
\begin{aligned}
\int\limits_{Q_3}\alpha[u][v]\,dQ_3 &= -\int\limits_{Q_3}\frac{\alpha}{b}[v_t][v]\,dQ_3=\int\limits_\gamma\frac{\alpha}{2b}[v]^2\Big|_{t=0}\,d\gamma+\int\limits_{Q_3}\frac{\alpha(-b')}{2b^2}[v]^2\,dQ_3\\
&\geqslant c^{-1}\int\limits_\gamma\Bigg(\int\limits_0^T b[u]\,d\tau\Bigg)^2 d\gamma+c^{-1}\int\limits_{Q_3}\Bigg(\int\limits_T^t b[u]\,d\tau\Bigg)^2 dQ_3.
\end{aligned}
$$

We conclude that $\langle\bar{\mathcal L}x,y\rangle_{Y^*\times Y}\geqslant c^{-1}\|x\|^2_{X_1}$. Then, by the Cauchy–Schwarz inequality, $\|\bar{\mathcal L}x\|_{Y^*}\cdot\|y\|_Y\geqslant c^{-1}\|x\|^2_{X_1}$.

We now show that $\|y\|_Y\leqslant c\|x\|_{X_1}$. Indeed, since $\vec\eta=\mathbf K\operatorname{grad}v$, we have

$$\begin{aligned}\|y\|_Y^2 &= \|v_t\|_{L_2(Q)}^2 + \sum_{k=1}^{2}\sum_{i=1}^{n}\|v_{\xi_i}\|_{L_2(Q_k)}^2 + \|\vec{\eta}\|_{L_2^n(Q)}^2 \\ &\leqslant \|v_t\|_{L_2(Q)}^2 + c\sum_{k=1}^{2}\sum_{i=1}^{n}\|v_{\xi_i}\|_{L_2(Q_k)}^2 \\ &\leqslant c\int_Q u^2\,dQ + c\sum_{k=1}^{2}\sum_{i=1}^{n}\int_{Q_k}\left(\int_T^t b u_{\xi_i}\,d\tau\right)^2 dQ_k \\ &\leqslant c\|x\|_{X_1}^2.\end{aligned}$$

Therefore,

$$\|\bar{\mathcal{L}}x\|_{Y^*} \geqslant c^{-1}\|x\|_{X_1}, \quad \forall x \in X. \qquad \square$$

A similar inequality for the adjoint operator can be proved in the same way.

Lemma 3. *There exists a positive constant $c > 0$ such that for all $y \in Y$, the inequality*

$$c^{-1}\|y\|_{Y_1} \leqslant \left\|\bar{\mathcal{L}}^+ y\right\|_{X^*}$$

is valid.

To prove this inequality, consider $\bar{\mathcal{L}}^+ y$ at the point $x = (u, \vec{\omega}) = \bar{I} y$, where

$$u = \int_0^t a(\tau)v(\tau,\xi)\,d\tau, \qquad \vec{\omega} = -\mathbf{K}\operatorname{grad} u.$$

In [24,25,36,37], similar inequalities are considered, but $\|x\|_{X_1}$ and $\|y\|_{Y_1}$ are semi-norms here.

4. Optimization of the parabolic system with discontinuity

Let an optimal control h of the parabolic system $\bar{\mathcal{L}}x = F(h)$ with insertions be defined as a minimum point of a functional $\mathcal{J}(h) = \Phi(u(t,\xi;h),h)$, where h is a control from an allowable set U_{ad} of a Banach space V of controls, $u(t,\xi;h)$ is a solution of $\bar{\mathcal{L}}x = F(h)$, $x = (u,\vec{\omega})$, and $F: V \to Y^*$ is a control function. Denote by U^* the set of optimal controls $h^* \in U_{ad}$.

In order to define the functional Φ correctly, one must guarantee the existence of a unique solution of $\bar{\mathcal{L}}x = F(h)$ for all $h \in U_{ad}$. This would follow from $F(U_{ad}) \subset R(\bar{\mathcal{L}})$, where $R(\bar{\mathcal{L}})$ is the range of $\bar{\mathcal{L}}$. However, the problem of describing the functional set of $R(\bar{\mathcal{L}})$ is very difficult. Moreover, in many important cases the inclusion $F(U_{ad}) \subset R(\bar{\mathcal{L}})$ is not valid at all.

To resolve these difficulties, we prove the existence and uniqueness for solutions of $\bar{\mathcal{L}}x = F$ in the natural sense and in a certain generalized sense for sufficiently wide sets of the right-hand sides $F \in Y^*$. This itself is of some interest.

Lemma 4. *The operators $\bar{\mathcal{L}}$ and $\bar{\mathcal{L}}^+$ are injective.*

Proof. Suppose that there exists $x = (u, \vec{\omega}) \in X$, $\bar{\mathcal{L}}x = 0$ in Y^*. Then $\langle \bar{\mathcal{L}}x, y\rangle_{Y^*\times Y} = 0$ for all $y \in Y$, including $y = Ix$ defined in Lemma 2. Applying the inequality from Lemma 2, we obtain $0 = \langle \bar{\mathcal{L}}x, y\rangle_{Y^*\times Y} \geqslant c^{-1}\|x\|^2_{X_1}$. Therefore, $u = 0$ in $L_2(Q)$, and thus in $W_2^{1,1/1}(Q)$. Hence, the equality $\langle \bar{\mathcal{L}}x, y\rangle_{Y^*\times Y} = 0$ can be rewritten as

$$\sum_{k=1}^{2}\sum_{i=1}^{n}\int_{Q_k} \omega_i \left(\sum_{j=1}^{n} \bar{k}_{ij}\eta_j - \frac{\partial v}{\partial \xi_i}\right) dQ_k = 0, \quad \forall y = (v, \bar{\eta}) \in Y.$$

For $y = (0, \mathbf{K}\vec{\omega})$ the equality takes the form $\|\vec{\omega}\|^2_{L_2^n(Q)} = 0$. Hence, $\vec{\omega} = \vec{0}$ in $L_2^n(Q)$.

Injectivity of $\bar{\mathcal{L}}^+$ can be shown in the same way. □

Theorem 2. *For an arbitrary $F \in S_1 = \{(f, \vec{0}) \mid f \in Y_1^*\} \subset Y^*$, there exists a unique element $x \in X$ such that $\bar{\mathcal{L}}x = F$ in Y^*.*

Proof. In view of Lemma 3, for an arbitrary $y \in Y$, we have

$$\left|\langle F, y\rangle_{Y^*\times Y}\right| = \left|\langle f, v\rangle_{Y_1^*\times Y_1}\right| \leqslant \|f\|_{Y_1^*}\|v\|_{Y_1} = \|f\|_{Y_1^*}\|y\|_{Y_1} \leqslant c\left\|\bar{\mathcal{L}}^+ y\right\|_{X^*}.$$

By injectivity of $\bar{\mathcal{L}}^+$, the expression $\langle F, y\rangle_{Y^*\times Y}$ defines a continuous linear functional $\mu(\bar{\mathcal{L}}^+ y) = \langle F, y\rangle_{Y^*\times Y}$ in X^*. Using Banach theorem, extend the functional from set $R(\bar{\mathcal{L}}^+)$ to the whole space X^*.

Since $(X^*)^* = X$, there exists an element $x \in X$ such that

$$\mu(\bar{\mathcal{L}}^+ y) = \langle F, y\rangle_{Y^*\times Y} = \langle x, \bar{\mathcal{L}}^+ y\rangle_{X\times X^*}$$

for all $y \in Y$. Therefore, $\langle \bar{\mathcal{L}}x, y\rangle_{Y^*\times Y} = \langle F, y\rangle_{Y^*\times Y}$, or $\bar{\mathcal{L}}x = F$ in Y^*.

Uniqueness of the solution follows from the injectivity of $\bar{\mathcal{L}}$. □

Corollary 1. *Parabolic system* (1)–(3) *in a domain with insertions has a unique solution $u \in W_2^{1,1/1}(Q)$ for all $f \in L_2(Q)$.*

Corollary 2. *The equality $\{g \in X_1^* \mid (g, \vec{0}) \in R(\bar{\mathcal{L}}^+)\} = X_1^*$ holds.*

Definition 1. A function $u \in X_1$ is called a generalized solution of the equation $\bar{\mathcal{L}}x = F$ if there exists a sequence $x_k \in X$ such that

$$\left\|(u, \vec{0}) - x_k\right\|_{X_1} \to 0, \qquad \|F - \bar{\mathcal{L}}x_k\|_{Y^*} \to 0, \quad k \to \infty.$$

Theorem 3. *For an arbitrary $F \in S_2 = \{(f, \vec{0}) \mid f \in W_{2,*}^{-1,1/1}(Q)\} \subset Y^*$, there exists a unique generalized solution $u \in X_1$ of $\bar{\mathcal{L}}x = F$.*

Proof. The set S_1 is dense in S_2, where S_1 and S_2 are considered as the subsets of Y^*. Hence there exists a sequence $F_k \in S_1$ such that $F_k \to F$ in Y^*, as $k \to \infty$. By Theorem 2, there exists a sequence $x_k \in X$ such that $\bar{\mathcal{L}}x_k = F_k$ and, due to inequality from Lemma 2, the sequence x_k

is fundamental with respect to the semi-norm $\|\cdot\|_{X_1}$. Thus, there exists an element $u \in X_1$ such that $\|(u, \vec{0}) - x_k\|_{X_1} \to 0$, i.e. u is a generalized solution of the equation $\bar{\mathcal{L}}x = F$.

If $\bar{u} \in X_1$ is another generalized solution, then we have

$$\|u - \bar{u}\|_{X_1} \leqslant \|u_k - \bar{u}_k\|_{X_1} + o(1) \leqslant c\big\|\bar{\mathcal{L}}x_k - \bar{\mathcal{L}}\bar{x}_k\big\|_{Y^*} + o(1) = o(1). \qquad \square$$

Corollary 3. *Parabolic system* (1)–(3) *in a domain with insertions has a unique solution* $u \in L_2(Q)$ *for all* $f \in W_{2,*}^{-1,1/1}(Q)$.

Corollary 4. *There exists a constant* $c > 0$ *such that for all* $F \in S_2$ *the inequality* $\|u\|_{X_1} \leqslant c\|F\|_{Y^*}$ *holds, where* u *is a generalized solution of* $\bar{\mathcal{L}}x = F$.

In view of the above discussion, below we will study a minimization problem with the functional $\mathcal{J}(h) = \Phi(u(t, \xi; h), h)$, $h \in U_{ad}$, and the control function $F : V \to Y^*$, where $R(F) \subset S_2$. Here $X_1 \times V$ constitutes $D(\Phi)$, the domain of Φ.

Note that one could make a similar investigation for the functional Φ with $D(\Phi) = X \times V$ and the control function $F : V \to Y^*$ ($R(F) \subset S_1$).

The following is a useful fact about the generalized solution of $\bar{\mathcal{L}}x = F$.

Lemma 5. *For a function* $u \in X_1$ *to be a generalized solution of* $\bar{\mathcal{L}}x = F$, *it is necessary* (*and sufficient if* $F \in S_2$) *that for all* $y \in Y$, $\bar{\mathcal{L}}^+ y = (g, \vec{0})$, $g \in X_1^*$, *the equality* $\langle u, g\rangle_{X_1 \times X_1^*} = \langle F, y\rangle_{Y^* \times Y}$ *is valid.*

Proof. Let u be a generalized solution of $\bar{\mathcal{L}}x = F$ and let $x_k = (u_k, \vec{\omega}_k) \in X$ be the corresponding sequence. Then

$$\langle u_k, g\rangle_{X_1 \times X_1^*} = \big\langle x_k, \bar{\mathcal{L}}^+ y\big\rangle_{X \times X^*} = \langle \bar{\mathcal{L}}x_k, y\rangle_{Y^* \times Y}$$

for all $y \in Y$ such that $\bar{\mathcal{L}}^+ y = (g, \vec{0})$, where $g \in X_1^*$.

Passing to the limit as $k \to \infty$, we obtain the desired equality.

Conversely, suppose that the equality $\langle u, g\rangle_{X_1 \times X_1^*} = \langle F, y\rangle_{Y^* \times Y}$ holds for all $y \in Y$ and $g \in X_1^*$ such that $\bar{\mathcal{L}}^+ y = (g, \vec{0})$. By Theorem 3, equation $\bar{\mathcal{L}}x = F$ has a generalized solution $u^* \in X_1$, therefore $\langle u - u^*, g\rangle_{X_1 \times X_1^*} = 0$. Because of Corollary 2, the function $g \in X_1^*$ attains all the elements of space X_1^* as values, hence $u = u^*$. $\square$

5. Existence of the optimal control of a parabolic system with an insertion and properties of the functional $\mathcal{J}$

Theorem 4. *Suppose the following holds*:

(1) Φ *is a weakly lower semi-continuous functional in the space* $X_1 \times V$;
(2) U_{ad} *is a weakly compact set in a Banach space* V;
(3) $F : V \to Y^*$ ($R(F) \subset S_2$) *is a weakly continuous control function* (*if* $h_k \to h^*$ *weakly in* V *then* $F(h_k) \to F(h^*)$ *weakly in* Y^*).

Then there exists an optimal control of the system $\bar{\mathcal{L}}x = F(h)$.

Proof. By virtue of the weak compactness of the set U_{ad}, there exists a weakly convergent sequence of controls $h_k \xrightarrow{w} h^* \in U_{ad}$ which minimizes the functional $\mathcal{J}$. Thus, $F(h_k)$ converges to $F(h^*)$ weakly in Y^*. Hence, by the inequality $\|u\|_{X_1} \leqslant c\|F\|_{Y^*}$ (Corollary 4), boundedness $u(h_k)$ in the space X_1 follows immediately, where $u(h_k)$ is a sequence of generalized solutions of $\bar{\mathcal{L}}x = F(h_k)$. Since X_1 is a reflexive space, a closed, convex and bounded set in X_1 is weakly compact. Therefore, the sequence $u(h_k)$ contains a weakly convergent subsequence $u(h_{k_m}) \xrightarrow{w} u^* \in X_1$.

Let $u(h^*)$ be a generalized solution of $\bar{\mathcal{L}}x = F(h^*)$. We prove that $u^* = u(h^*)$ in X_1. By Lemma 5, we have

$$\langle u(h_{k_m}), g\rangle_{X_1 \times X_1^*} = \langle F(h_{k_m}), y\rangle_{Y^* \times Y}, \quad \forall y \in Y : \bar{\mathcal{L}}^+ y = (g, \vec{0}), \quad g \in X_1^*.$$

Passing to the limit as $m \to \infty$, we obtain

$$\langle u^*, g\rangle_{X_1 \times X_1^*} = \langle F(h^*), y\rangle_{Y^* \times Y}, \quad \forall y \in Y : \bar{\mathcal{L}}^+ y = (g, \vec{0}), \quad g \in X_1^*.$$

On the other hand, again by Lemma 5, we have $u^* = u(h^*)$.

Taking into consideration the weak lower semi-continuity of Φ, we obtain

$$\inf_{h \in U_{ad}} \mathcal{J}(h) = \varliminf_{m \to \infty} \Phi(u(h_{k_m}), h_{k_m}) \geqslant \Phi(u(h^*), h^*).$$

Thus, $h^* \in U_{ad}$ is an optimal control. $\square$

Remark 2. Since the control function F may be non-linear, and the functional Φ may be non-convex, optimal control may not be unique.

Remark 3. It follows from the proof that an arbitrary minimizing sequence of controls h_k converges to the optimal control set U^* weakly in V, i.e. for all $l \in V^*$ we have

$$\inf_{h^* \in U^*} |l(h_k - h^*)| \to 0, \quad k \to \infty.$$

Analogously, if U_{ad} is a compact set, then one can prove that an arbitrary minimizing sequence h_k converges to the set U^*, that is, $\rho(h_k, U^*) \to 0$.

Remark 4. Considering the inequality from Corollary 4 and assuming the control function F and the functional Φ are smooth, one could investigate various stability properties of the optimal system or the functional $\mathcal{J}$ with respect to control disturbance. For instance, if $F : V \to Y^*$ and $\Phi : X_1 \times V \to \mathbb{R}$ are continuous functions, then $\mathcal{J} : V \to \mathbb{R}$ is a continuous functional. Therefore, the optimal control problem is stable with respect to control disturbance.

If there exist the Fréchet derivative $\Phi' = (\Phi'_u, \Phi'_h)$ of the functional $\Phi : X_1 \times V \to \mathbb{R}$ at a point $(u(h), h)$ and the Fréchet derivative $F'(h)$ of $F : V \to Y^*$ at a point $h \in U_{ad}$, then one can consider the differential properties of $\mathcal{J} : V \to \mathbb{R}$, which enables the study of the gradient methods in order to solve the optimization problem.

Theorem 5. *If there exist the Fréchet derivatives $\Phi'(u(h),h) = (\Phi'_u, \Phi'_h)$ and $F'(h)$, then there exists the Fréchet derivative $\mathcal{J}'(h)$ of $\mathcal{J}$ defined as follows*:

$$\langle \mathcal{J}'(h), \Delta h\rangle_{V^*\times V} = \langle F'(h)(\Delta h), y(h)\rangle_{Y^*\times Y} + \langle \Phi'_h(u(h),h), \Delta h\rangle_{V^*\times V}, \tag{17}$$

where functions $u(h) \in X_1$, $y(h) \in Y$ are the solutions of the operator equations $\bar{\mathcal{L}}x = F(h)$, $\bar{\mathcal{L}}^+ y = (\Phi'_h(u(h),h), \vec{0})$.

Proof. Finding the linear part of the increment $\mathcal{J}(h+\Delta h) - \mathcal{J}(h)$, we have

$$\mathcal{J}(h+\Delta h) - \mathcal{J}(h) = \langle \Phi'_u(u(h),h), \Delta u\rangle_{X_1^*\times X_1} + \langle \Phi'_h(u(h),h), \Delta h\rangle_{V^*\times V} + o,$$

where $o = o(\|(\Delta u, \Delta h)\|_{X_1\times V})$ and $\Delta u = u(h+\Delta h) - u(h) \in X_1$ is the generalized solution of the equation $\bar{\mathcal{L}}x = \Delta F$ with the right-hand side $\Delta F = F(h+\Delta h) - F(h) \in Y^*$. Applying Lemma 5, we have

$$\langle \Phi'_u, \Delta u\rangle_{X_1^*\times X_1} = \langle \Delta F(h), y(h)\rangle_{Y^*\times Y} = \langle F'(h)(\Delta h), y(h)\rangle_{Y^*\times Y} + o(\|\Delta h\|).$$

Using the corollary of Theorem 3, we obtain

$$\|\Delta x\|_{X_1} \leqslant c\|\Delta F\|_{Y^*} \leqslant c\|F'(h)(\Delta h)\|_{Y^*} + o(\|\Delta h\|_V) = O(\|\Delta h\|_V). \qquad \square$$

Using relations (17) and inequalities from Lemmas 1 and 2, one can investigate various properties of the gradient of $\mathcal{J}$ (uniform continuity, uniform or local Lipschitz condition, etc.) which depend on the smoothness of the control function F and of the functional Φ.

To illustrate this, let us consider the following theorem of this type.

Theorem 6. *If the Fréchet derivative $F'(h)$ $(R(F) \subset S_2)$ is α-Hölder continuous in a bounded neighborhood $U \subset U_{ad}$ (i.e. there exist $c > 0$ and $\alpha \in (0,1]$ such that for all $h_1, h_2 \in U$ the inequality $\|F'(h_1) - F'(h_2)\| \leqslant c\|h_1 - h_2\|_V^\alpha$ holds), and the derivatives $\Phi'_u(u,h)$, $\Phi'_h(u,h)$ satisfy the same condition on $U_1 \times U$ (here U_1 is a corresponding domain of $u(h)$), then the Fréchet derivative $\mathcal{J}'(h)$ is α-Hölder continuous on U.*

Proof. Since the Fréchet derivative F' is α-Hölder continuous in the bounded domain U, so F' is bounded in U. Then we have

$$\begin{aligned}
&\left|\langle \mathcal{J}'(h_1) - \mathcal{J}'(h_2), \Delta h\rangle_{V^*\times V}\right| \\
&\quad\leqslant \|\Phi'_h(u(h_1),h_1) - \Phi'_h(u(h_2),h_2)\|_{V^*}\|\Delta h\|_V \\
&\qquad + \left|\langle F'(h_2)(\Delta h), y(h_1) - y(h_2)\rangle_{Y^*\times Y}\right| + \left|\langle (F'(h_1) - F'(h_2))(\Delta h), y(h_1)\rangle_{Y^*\times Y}\right| \\
&\quad\leqslant c\|(u(h_1) - u(h_2), h_1 - h_2)\|^\alpha_{X_1\times V}\|\Delta h\|_V + \left|\langle F'(h_2)(\Delta h), y(h_1) - y(h_2)\rangle_{Y^*\times Y}\right| \\
&\qquad + \left|\langle (F'(h_1) - F'(h_2))(\Delta h), y(h_1)\rangle_{Y^*\times Y}\right|.
\end{aligned}$$

Let us estimate $|\langle F'(h_2)(\Delta h), y(h_1) - y(h_2)\rangle_{Y^*\times Y}|$. It is clear that $F'(h_2)(\Delta h) \in S_2$. By Lemma 5 and Theorem 3, there exists a generalized solution $u(h_2, \Delta h) \in X_1$ of $\bar{\mathcal{L}}x = F'(h_2)(\Delta h)$ such that

$$\big\langle F'(h_2)(\Delta h), y(h_1) - y(h_2)\big\rangle_{Y^*\times Y} = \big\langle u(h_2, \Delta h), \Phi'_u\big(u(h_1), h_1\big) - \Phi'_u\big(u(h_2), h_2\big)\big\rangle_{X_1\times X_1^*}.$$

Then

$$\begin{aligned}
&\big|\big\langle F'(h_2)(\Delta h), y(h_1) - y(h_2)\big\rangle_{Y^*\times Y}\big| \\
&\quad\leqslant \big\|u(h_2, \Delta h)\big\|_{X_1}\big\|\Phi'_u\big(u(h_1), h_1\big) - \Phi'_u\big(u(h_2), h_2\big)\big\|_{X_1^*} \\
&\quad\leqslant c\big\|u(h_2, \Delta h)\big\|_{X_1}\big\|\big(u(h_1) - u(h_2), h_1 - h_2\big)\big\|^{\alpha}_{X_1\times V}.
\end{aligned}$$

A similar argument yields

$$\begin{aligned}
&\big|\big\langle\big(F'(h_1) - F'(h_2)\big)(\Delta h), y(h_1)\big\rangle_{Y^*\times Y}\big| \\
&\quad\leqslant \big\|u(h_1, h_2, \Delta h)\big\|_{X_1}\big\|\Phi'_u\big(u(h_1), h_1\big)\big\|_{X_1^*} \\
&\quad\leqslant c\big\|\big(F'(h_1) - F'(h_2)\big)(\Delta h)\big\|_{Y^*}\big\|\Phi'_u\big(u(h_1), h_1\big)\big\|_{X_1^*} \\
&\quad\leqslant c_1\|h_1 - h_2\|^{\alpha}_V\|\Delta h\|_V\big\|\Phi'_u\big(u(h_1), h_1\big)\big\|_{X_1^*},
\end{aligned}$$

where $u(h_1, h_2, \Delta h) \in X_1$ is the generalized solution of $\bar{\mathcal{L}}x = (F'(h_1) - F'(h_2))(\Delta h)$.

To complete the proof, one should take into account the following inequalities

$$\begin{aligned}
&\big\|u(h_1) - u(h_2)\big\|_{X_1} \leqslant c\big\|F(h_1) - F(h_2)\big\|_{Y^*} \leqslant c_1\|h_1 - h_2\|_V, \\
&\big\|u(h_2, \Delta h)\big\|_{X_1} \leqslant c\big\|F'(h_2)(\Delta h)\big\|_{Y^*} \leqslant c_1\|\Delta h\|_V, \\
&\big\|\Phi'_u\big(u(h_1), h_1\big)\big\|_{X_1^*} \leqslant \big\|\Phi'_u\big(u(h_1), h_1\big) - \Phi'_u\big(u(h_0), h_0\big)\big\|_{X_1^*} \\
&\qquad\qquad + \big\|\Phi'_u\big(u(h_0), h_0\big)\big\|_{X_1^*} \leqslant c\|h_1 - h_0\|^{\alpha}_V + c \leqslant c_1,
\end{aligned}$$

where $h_0 \in U \subset U_{ad}$ is a fixed point. □

Definition 2. A system $\bar{\mathcal{L}}x = F$ is called asymptotically controllable in a Banach space E by a set of admissible controls U_{ad} if for an arbitrary element $u^* \in E$, there exists a sequence of controls $h_i \in U_{ad}$ such that $\|u(h_i) - u^*\|_E \to 0$ as $i \to 0$, and $u(h_i)$ are solutions of $\bar{\mathcal{L}}x = F(h_i)$.

Theorem 7. *If the set $F(U_{ad}) \subset S_2$ is dense in S_2, where $F(U_{ad})$ and S_2 are considered as subsets of Y^*, the system $\bar{\mathcal{L}}x = F$ is asymptotically controllable in X_1 by the set of admissible controls U_{ad}.*

Proof. Let u^* be an arbitrary element of the space X_1. The set $C^1_{bd}(\bar{Q}_1, \bar{Q}_2)$ is dense in X_1, hence there exists a sequence $u_i \in C^1_{bd}(\bar{Q}_1, \bar{Q}_2)$ such that $\|u_i - u^*\|_{X_1} \to 0$ as $i \to \infty$.

Let $x_i = (u_i, -\mathbf{K}\operatorname{grad} u_i) \in X$. It is easy to prove that $\bar{\mathcal{L}}x_i \in S_2$.

Since $F(U_{ad})$ is dense in S_2, for every fixed positive integer i, there exists a sequence of controls $h_{i,k} \in U_{ad}$ such that $\|\bar{\mathcal{L}}x_i - F(h_{i,k})\|_{Y^*} = \varepsilon_k$, where $\varepsilon_k \to 0$. Using the inequality proved above, we obtain

$$\begin{aligned}\left\|u^* - u(h_{i,k})\right\|_{X_1} &\leqslant \left\|u^* - u_i\right\|_{X_1} + \left\|u_i - u(h_{i,k})\right\|_{X_1} \\ &\leqslant \left\|u^* - u_i\right\|_{X_1} + c\left\|\bar{\mathcal{L}}x_i - F(h_{i,k})\right\|_{Y^*} = \left\|u^* - u_i\right\|_{X_1} + c\varepsilon_k \to 0\end{aligned}$$

as $k \to \infty$, $i \to \infty$. □

6. Applications to the problem of pulse optimal control

In this section we consider applications of our results to a specific problem of pulse optimal control of the parabolic system with insertions.

Let an optimal control h of the parabolic system $\bar{\mathcal{L}}x = F(h)$ be a minimum point of a functional $\mathcal{J}^*(h) = \Phi^*(u(t,\xi;h),h)$, where

$$\Phi^*(u,h) = \int_Q (u-u^*)^2\,dQ + \int_{Q_3}\left(\int_T^t b[u]\,d\tau - u^{**}\right)^2 dQ_3 + \|h-h^*\|_V^2,$$

and $u^* \in L_2(Q)$, $u^{**} \in L_2(Q_3)$, $h^* \in V$ are certain elements which characterize the desired modes of operation of the system in the domain Q, on the surface Q_3, and the desired optimal control.

Since $\|\cdot\|$ is a weakly lower semicontinuous functional, Φ^* is a weakly lower semi-continuous functional in $X_1 \times V$.

For an arbitrary $v \in W_{2,*}^{1,1/1}(Q)$ the following inequality

$$\left|\int_\Omega v(t^*,\xi)\,d\Omega\right| = \left|\int_\Omega \int_T^{t^*} v_t(\tau,\xi)\,d\tau\,d\Omega\right| \leqslant c\left(\int_Q v_t^2\,dQ\right)^{1/2} \leqslant c\|v\|_{W_{2,*}^{1,1/1}(Q)}$$

is valid, hence $\delta(t-t^*)(v) = \int_\Omega v(t^*,\xi)\,d\Omega \in W_{2,*}^{-1,1/1}(Q)$.

Therefore one may consider a pulse control problem. Let the control function $F(h)$ be of the pulse form

$$F(h) = \big(f(h),\vec{0}\big) = \left(\sum_{s=1}^N \delta(t-t_s)\varphi_s(\xi),\vec{0}\right) \in S_2 \subset Y^*,$$

where $\delta(t-t_s)$ is the delta function, $\vec{t} = (t_1,\dots,t_N) \in \mathbb{R}^N$ is a vector of pulse time points, $\vec{\varphi} = (\varphi_1,\dots,\varphi_N) \in L_2^N(\Omega)$ is a vector of pulse source strengths, $h = (\vec{t},\vec{\varphi}) \in U_{ad} \subset V$ is a control, $V = \mathbb{R}^N \times L_2^N(\Omega)$, and $U_{ad} = \{h \in V \mid t_s \in [0,T],\ \|\varphi_s\|_{L_2(\Omega)} \leqslant C\}$. It is clear that U_{ad} is a weakly compact set in the Hilbert space V.

In order to apply the obtained results to the system $\bar{\mathcal{L}}x = F(h)$ with the pulse control function, one must show the weak continuity of F (Theorem 4), prove existence and find the Fréchet derivative (Theorem 5), and study the gradient smoothness (Theorem 6).

We now show that $f: V \to W_{2,*}^{-1,1/1}(Q)$ is a weakly continuous function. Indeed, let $h^k \to h^*$ be a weakly convergent sequence in V as $k \to \infty$, where $h^k = (\vec{t}^k, \vec{\varphi}^k) \in U_{ad}$ and $h^* = (\vec{t}^*, \vec{\varphi}^*) \in U_{ad}$. Because of the weak convergence of $\vec{\varphi}^k \to \vec{\varphi}^*$ in $L_2^N(\Omega)$, we have

$$\begin{aligned}
&\langle f(h^k), v\rangle_{W_{2,*}^{-1,1/1} \times W_{2,*}^{1,1/1}} \\
&\quad = \sum_{s=1}^N \int_\Omega v(t_s^k, \xi)\varphi_s^k(\xi)\, d\Omega \\
&\quad = \sum_{s=1}^N \int_\Omega v(t_s^*, \xi)\varphi_s^k(\xi)\, d\Omega + \sum_{s=1}^N \int_\Omega \Delta v \varphi_s^k(\xi)\, d\Omega \to \langle f(h^*), v\rangle_{W_{2,*}^{-1,1/1} \times W_{2,*}^{1,1/1}}, \\
&\quad k \to \infty,\ \forall v \in W_{2,*}^{1,1/1},
\end{aligned}$$

where $\Delta v = v(t_s^k, \xi) - v(t_s^*, \xi) \in W_{2,*}^{1,1/1}(Q)$. This follows from

$$\left|\sum_{s=1}^N \int_\Omega \Delta v \varphi_s^k(\xi)\, d\Omega\right| \leqslant \sum_{s=1}^N \left|\int_\Omega \int_{t_s^*}^{t_s^k} v_t(\tau, \xi)\varphi_s^k(\xi)\, d\tau\, d\Omega\right| \leqslant c \sum_{s=1}^N \left|t_s^k - t_s^*\right|^{1/2} \to 0.$$

Thus, the weak continuity of $f: V \to W_{2,*}^{-1,1/1}(Q)$ has been proved, and there exists an optimal control of the parabolic system (1)–(3) with insertions.

One can easily find the gradient of the control function f with respect to $\vec{\varphi}$ at a point $h^* = (\vec{t}^*, \vec{\varphi}^*)$:

$$\langle f'_{\vec{\varphi}}(h^*)(\Delta\vec{\varphi}), v\rangle_{W_{2,*}^{-1,1/1} \times W_{2,*}^{1,1/1}} = \sum_{s=1}^N \int_\Omega v(t_s^*, \xi)\Delta\varphi_s\, d\Omega, \quad \Delta\vec{\varphi} \in L_2^N(\Omega)$$

and prove that the gradient is α-Hölder continuous with $\alpha = 1/2$.

Thus, one can try to apply numerical methods to find the optimal vector of pulse source strengths $\vec{\varphi}^*(\xi)$. However, with respect to $\vec{t}$, the control function $f: V \to W_{2,*}^{-1,1/1}(Q)$ does not have a Fréchet derivative. To find an optimal vector of pulse points of time $\vec{t}^*$, we investigate a regularization of the control function $F(h) = (f(h), \vec{0})$.

Consider a family of the regularized control functions $F_\varepsilon(h) = (f_\varepsilon(h), \vec{0})$, $f_\varepsilon(h) \in P$, $P \subset W_{2,*}^{-1,1/1}(Q)$ and an optimization problem with a functional $\mathcal{J}_\varepsilon(h) = \Phi(u_\varepsilon(h), h)$ on the admissible set $U_{ad} \subset V$, where $u_\varepsilon(h)$ is the generalized solution of the regularized problem $\bar{\mathcal{L}}x = F_\varepsilon(h)$. Let U_ε^* be the set of optimal controls of the regularized optimization problem.

The following general theorem shows the relation between the solutions of the initial optimization problem and the solutions of the regularized one.

Theorem 8. *Suppose the following holds*:

(1) *the admissible set of controls U_{ad} is weakly compact in the Banach space V*;

(2) *the control functions* $F(h)$ *and* $F_\varepsilon(h)$ *satisfy the following conditions*:
 (a) *if* $h_{\varepsilon_k} \to h$ *weakly in* V *then* $F_{\varepsilon_k}(h_{\varepsilon_k}) \to F(h)$ *weakly in* Y^* *for an arbitrary* $\varepsilon_k \to 0$,
 (b) F, F_{ε_k} *are weakly continuous control functions with respect to* h *in* Y^*,
 (c) $\|F_\varepsilon(h) - F(h)\|_{Y^*} \to 0$ *as* $\varepsilon \to 0$ *for all* $h \in U_{ad} \subset V$;
(3) *the functional* $\Phi(u, h)$ *is weakly lower semicontinuous in* $X_1 \times V$ *and strongly upper semicontinuous with respect to* u *in* $X_1 \times V$.

Then the problems of optimal control for $\bar{\mathcal{L}}x = F(h)$ *and* $\bar{\mathcal{L}}x = F_\varepsilon(h)$ *have solutions and the sets* U_ε^* *converge weakly to* U^* *as* $\varepsilon \to 0$, *i.e. for all* $l \in V^*$ *and* $h_\varepsilon^* \in U_\varepsilon^*$ *we have* $\inf_{h^* \in U^*} |l(h_\varepsilon^* - h^*)| \to 0$ *as* $\varepsilon \to 0$.

Proof. Taking into account conditions (1), (2)(b), (3), by Theorem 4, we have $U_\varepsilon^* \neq 0$, $U^* \neq 0$.

We will prove that an arbitrary sequence $h_{\varepsilon_k}^* \in U_{\varepsilon_k}^*$ ($\varepsilon_k \to 0$) contains a weakly convergent subsequence that converges to $h^* \in U^*$ weakly in V, and the theorem will follow.

By weak compactness of U_{ad}, the sequence $h_{\varepsilon_k}^* \in U_{\varepsilon_k}^*$ contains a weakly convergent subsequence $U_{\varepsilon_m}^* \ni h_{\varepsilon_m}^* \xrightarrow{w} h^* \in U_{ad}$. Consider the sequence of the generalized solutions $u_{\varepsilon_m}(h_{\varepsilon_m}^*)$. We have $\|u_{\varepsilon_m}(h_{\varepsilon_m}^*)\|_{X_1} \leqslant c\|F_{\varepsilon_m}(h_{\varepsilon_m}^*)\|_{Y^*}$. From condition (2)(a), we conclude that the sequence $F_{\varepsilon_m}(h_{\varepsilon_m}^*)$ is bounded in Y^*. Thus, there exists a weakly convergent subsequence $u_{\varepsilon_p}(h_{\varepsilon_p}^*)$ of $u_{\varepsilon_m}(h_{\varepsilon_m}^*)$, where $u_{\varepsilon_p}(h_{\varepsilon_p}^*) \xrightarrow{w} u \in X_1$. Considering that $u_{\varepsilon_p}(h_{\varepsilon_p}^*)$ are the generalized solutions of $\bar{\mathcal{L}}x = F_{\varepsilon_p}(h_{\varepsilon_p}^*)$, we have

$$\langle u_{\varepsilon_p}(h_{\varepsilon_p}^*), g\rangle_{X_1 \times X_1^*} = \langle F_{\varepsilon_p}(h_{\varepsilon_p}^*), y\rangle_{Y^* \times Y}, \quad \forall y \in Y,\ \bar{\mathcal{L}}^+ y = (g, \vec{0}), \quad g \in X_1^*.$$

Passing to the limit as $p \to \infty$, we obtain

$$\langle u, g\rangle_{X_1 \times X_1^*} = \langle F(h^*), y\rangle_{Y^* \times Y}, \quad \forall y \in Y,\ \bar{\mathcal{L}}^+ y = (g, \vec{0}), \quad g \in X_1^*.$$

Applying Lemma 5, we infer that $u = u(h^*)$ is the generalized solution of $\bar{\mathcal{L}}x = F(h^*)$.

Since the functional $\Phi(u, h)$ is weakly lower semicontinuous, we have

$$\mathcal{J}(h^*) = \Phi(u(h^*), h^*) \leqslant \varliminf_{p\to\infty} \Phi(u_{\varepsilon_p}(h_{\varepsilon_p}^*), h_{\varepsilon_p}^*) = \varliminf_{p\to\infty} \mathcal{J}_{\varepsilon_p}(h_{\varepsilon_p}^*) \leqslant \varliminf_{p\to\infty} \mathcal{J}_{\varepsilon_p}(h),$$

for all $h \in U_{ad}$.

On the other hand, by inequality

$$\|u_{\varepsilon_p}(h) - u(h)\|_{X_1} \leqslant c\|F_{\varepsilon_p}(h) - F(h)\|_{Y^*} \underset{p\to\infty}{\to} 0,$$

we have the strong convergence of $u_{\varepsilon_p}(h) \to u(h)$ in X_1. Hence, for all $h \in U_{ad}$ we have

$$\mathcal{J}(h) = \Phi(u(h), h) \geqslant \varlimsup_{p\to\infty} \Phi(u_{\varepsilon_p}(h), h) = \varlimsup_{p\to\infty} \mathcal{J}_{\varepsilon_p}(h).$$

We conclude that $\mathcal{J}(h^*) \leqslant \mathcal{J}(h)$ for all $h \in U_{ad}$, i.e. $h^* \in U^*$. □

As for the problem of the pulse control, consider for instance the following regularization

$$f_\varepsilon(h) = \sum_{s=1}^{N} g_\varepsilon(t, t_s)\varphi_s(\xi) \in P = L_2(Q), \quad 0 < \varepsilon < T,$$

where

$$g_\varepsilon(t, t_s) = \begin{cases} 1/\varepsilon, & t \in I = [\theta_s, \varepsilon + \theta_s], \quad \theta_s = (T - \varepsilon)t_s/T. \\ 0, & t \in [0, T] \backslash I, \end{cases}$$

It is easy to show that this regularization satisfies conditions of the theorem.

The regularized control function F_ε has the Fréchet derivative, hence one can consider gradient methods to find an optimal vector of pulse points of time $\vec{t}^*$.

Since the weak and strong convergences are equivalent in $\mathbb{R}^n$, an optimal vector $\vec{t}_\varepsilon^*$ of pulse points of time of the regularized problem strongly converges to the optimal vector $\vec{t}^*$ of the original optimization problem.

Applying a pulse excitation, one could achieve the asymptotic controllability of the parabolic system. Indeed, it is not difficult to show that the set

$$\left\{ \sum_{s=1}^{N} c_s \delta(t - t_s)\varphi_s(\xi) \mid t_s \in [0, T],\ c_s \in \mathbb{R},\ N \in \mathbb{N},\ \varphi_s \in L_2(\Omega),\ \|\varphi_s\|_{L_2(\Omega)} = 1 \right\}$$

is dense in S_2, therefore, by Theorem 7, the system (1)–(3) is asymptotically controllable in X_1.

Acknowledgments

The author expresses sincere gratitude to referee of J. Differential Equations for critical remarks and useful suggestions concerning results of the article.

References

[1] P.A. Krutitskii, The mixed harmonic problem in a bounded cracked domain with Dirichlet condition on cracks, J. Differential Equations 198 (2) (2004) 422–441.
[2] J. Ferchichi, J. Zolesio, Shape sensitivity for the Laplace–Beltrami operator with singularities, J. Differential Equations 196 (2) (2004) 340–384.
[3] D.Yu. Lyubimov, L.G. Smirnov, Dirichlet and related problems for a composed space with annular slits at the interface in the axisymmetric case, Comput. Math. Math. Phys. 44 (8) (2004) 1366–1373.
[4] A.F. Albou, V.I. Zubov, V.A. Inyakin, Optimal control of the process of melting and solidification of a substance, Comput. Math. Math. Phys. 44 (8) (2004) 1291–1305.
[5] A.F. Albou, V.I. Zubov, Optimal control of the process of solidification, Comput. Math. Math. Phys. 44 (1) (2004) 34–44.
[6] P.A. Krutitskii, A.I. Sgibnev, Singularities of the solution gradient in the generalized jump problem for the Laplace equation outside a cut on the plane, Differential Equations 39 (9) (2003) 1225–1236.
[7] J. Huang, J. Zou, Some new a priori estimates for second-order elliptic and parabolic interface problems, J. Differential Equations 184 (2) (2002) 570–586.
[8] G. Bal, Transport through diffusive and nondiffusive regions, embedded objects, and clear layers, SIAM J. Appl. Math. 62 (5) (2002) 1677–1697.
[9] J. Yang, S. Kalliadasis, J.H. Merkin, S.K. Scott, Wave propagation in spatially distributed excitable media, SIAM J. Appl. Math. 63 (2) (2002) 485–509.

[10] V.F. Piven', Integral and integro–differential equations of the two-dimensional transmission problem for velocity fields on a time-dependent boundary, Differential Equations 38 (12) (2002) 1809–1814.
[11] V.A. Tsurko, Finite-difference methods for approximate solutions of two-dimensional parabolic equations with discontinuous coefficients, Differential Equations 38 (7) (2002) 1070–1072.
[12] E.V. Zakharov, S.I. Safronov, R.P. Tarasov, The problem of resonance scattering by a hole in an acoustically soft surface of revolution, Differential Equations 38 (9) (2002) 1239–1244.
[13] G.I. Shishkin, Grid approximation of a singularly perturbed parabolic reaction–diffusion equation with a fast-moving source, Comput. Math. Math. Phys. 42 (6) (2002) 788–801.
[14] P.A. Krutitskii, The Neumann problem in a 2-D exterior domain with cuts and singularities at the tips, J. Differential Equations 176 (1) (2001) 269–289.
[15] C. Lederman, N. Wolanski, Uniqueness in a two-phase free-boundary problem, Adv. Differential Equations 6 (12) (2001) 1409–1442.
[16] O.N. Komarenko, Self-adjoint operators generated by transmission problems with nonhomogeneous conjugation conditions, Ukrainian Math. J. 53 (12) (2001) 1607–1622.
[17] P.A. Krutitskii, The modified jump problem for the Helmholtz equation, Ann. Univ. Ferrara Sez. VII (N.S.) 47 (2001) 285–296.
[18] G. Bal, L. Ryzhik, Diffusion approximation of radiative transfer problems with interfaces, SIAM J. Appl. Math. 60 (6) (2000) 1887–1912.
[19] P.A. Krutitskii, The jump problem for the Helmholtz equation and singularities at the edges, Appl. Math. Lett. 13 (2000) 71–76.
[20] A.M. Khludnev, V.A. Kovtunenko, Analysis of Cracks in Solids, WIT Press, Boston, 2000.
[21] C.J.S. Alves, T. Ha-Duong, Inverse scattering for elastic plane cracks, Inverse Problems 15 (1999) 91–97.
[22] N. Nishimura, Crack determination problems, Theor. Appl. Mech. 46 (1997) 39–57.
[23] K. Bryan, L.F. Caudill, Jr., Solvability of a parabolic boundary value problem with internal jump condition, Rose-Hulman Technical Report MSTR 00-04, http://www.rose-hulman.edu/math/researchpubs/MSTechReps.php.
[24] V.V. Semenov, Solvability of a parabolic transmission problem with the condition of a generalized proper lumped source, Differential Equations 41 (6) (2005) 878–886.
[25] S.I. Lyashko, Generalized Optimal Control of Linear Systems with Distributed Parameters, Kluwer Academic, Boston, 2002.
[26] I.I. Lyashko, L.I. Demchenko, G.E. Mistetskij, Numerical Solution of Problems in Heat and Mass Exchange in Porous Media (Chislennoe Reshenie Zadach Teploi Massoperenosa v Poristykh Sredakh), Naukova Dumka, Kiev, 1991, in Russian.
[27] D.A. Nomirovskii, Approximate methods for solving the boundary value problem for a parabolic equation with inhomogeneous transmission conditions of nonideal contact type, Comput. Math. Math. Phys. 46 (6) (2006) 995–1006.
[28] D.A. Nomirovskii, Optimization of parabolic systems with distributional coefficients, Rep. Nat. Acad. Sci. Ukraine (12) (2000) 77–82.
[29] D.A. Nomirovskii, Generalized solvability and optimization of singular parabolic systems, Rep. Nat. Acad. Sci. Ukraine (10) (2003) 30–35.
[30] D.A. Nomirovskii, Generalized solvability of parabolic systems with heterogeneous transmission conditions of imperfect contacts, Differential Equations 40 (10) (2004) 1390–1399.
[31] I.V. Sergienko, V.S. Deineka, Optimal Control of Distributed Systems with Conjugation Conditions, Kluwer Academic, Boston, 2005.
[32] V.S. Deineka, I.V. Sergienko, V.V. Skopetskii, Models and Methods for Solving Problems with Conjugation Conditions (Modeli i Metody Resheniya Zadach s Usloviyami Sopryazheniya), Naukova Dumka, Kiev, 1998, in Russian.
[33] F. Kikuchi, Numerical analysis of a mixed finite element method for plate buckling problems, Institute of Space and Aeronautical Science, University of Tokyo, Report No. 584, September 1980, pp. 165–190.
[34] B. Mercier, J. Osborn, J. Rappaz, P.A. Raviart, Eigenvalue approximation by mixed and hybrid methods, Math. Comput. 36 (154) (1981) 427–453.
[35] E. Sanchez-Palencia, Non-Homogeneous Media and Vibration Theory, Springer-Verlag, New York, 1980.
[36] S.I. Lyashko, D.A. Nomirovskii, T.I. Sergienko, Trajectory and final controllability in hyperbolic and pseudohyperbolic systems with generalized actions, Cybernet. Systems Anal. 37 (5) (2001) 756–763.
[37] S.I. Lyashko, D.A. Nomirovskii, Generalized solution and optimal controls in systems describing the dynamic of a viscous fluid, Differential Equations 39 (1) (2003) 90–98.